Hypersurfaces with large automorphism groups

F 6/28/24

Birational Geometry Seminar

Joint with Louis Esser

$/\mathbb{C}$

<u>Question</u> X : smooth hypersurface of degree d in $\mathbb{P}^{n+1}_{\mathbb{C}}$

What is the largest possible automorphism group $\mathrm{Aut}(X)$?

<u>Thm</u> [Esser, L.]

X : sm. hypersurf. of degree d in $\mathbb{P}^{n+1}_{\mathbb{C}}$ s.t. $n \geq 1$ and $d \geq 3$

The Fermat hypersurface achieves the largest $|\mathrm{Lin}(X)|$ in dim. n, degree d, of order $(n+2)!\, d^{n+1}$,

except when (n,d) is

$(1,4)$, $(1,6)$, $(2,4)$, $(2,6)$, $(2,12)$, $(4,6)$, $(4,12)$.

Moreover, for any X such that $|\mathrm{Lin}(X)| \geq (n+2)!\, d^{n+1}$, X is isomorphic to the Fermat hypersurf. or to one of the exceptions above.

<u>Remark</u> This statement was already known for (n,d):

- $\underline{n=1}$ Pambianco, Hani
- $(2,3)$: Dolgachev-Duncan
- $(3,3)$: Wei-Yu
- $(3,5)$: Ogviso-Yu
- $(4,3)$: Laza-Zheng

<u>Note</u> We use Lin(X) instead of Aut(X) in our thm.

- φ is a linear automorphism of X if $\exists$ an autom. $\Psi : \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ s.t. $\Psi|_X = \varphi$.
- $\mathrm{Lin}(X) \leq \mathrm{Aut}(X)$
- If $n \geq 1$ and $d \geq 3$ and $(n,d) \neq (1,3)$ or $(2,4)$, then $\mathrm{Lin}(X) = \mathrm{Aut}(X)$.
  - $n \geq 3$: consequence of Grothendieck-Lefschetz Thm
  - $n = 1$ or 2

    $n = 2$ and $(n,d) \neq (2,4)$ then $\mathrm{Lin}(X) = \mathrm{Aut}(X)$ [Matsumura-Monsky]

$X : \{f=0\} \subset \mathbb{P}^{n+1}$

- $\mathrm{Lin}(f) = \{g \in GL_{n+2}(\mathbb{C}) \text{ s.t. } g\cdot f = f\}$.
- $|\mathrm{Lin}(f)| = d\,|\mathrm{Lin}(X)|$

The Fermat hypersurface: For any $n, d \in \mathbb{Z}_{>0}$, the Fermat hypersurface of dim n, deg d is $X_F \subset \mathbb{P}^{n+1}$

$$X_F : \quad x_0^d + x_1^d + \cdots + x_{n+1}^d = 0 \qquad \text{(smooth)}$$

- $\mathrm{Aut}(X)$ contains $(\mathbb{Z}/d)^{\oplus(n+1)} \rtimes S_{n+2}$

- [Shioda, Kontogeorgis] If $d \geq 3$, then $\mathrm{Lin}(X) = (\mathbb{Z}/d)^{\oplus n+1} \rtimes S_{n+2}$

$$\Rightarrow |\mathrm{Lin}(X)| = (n+2)!\, d^{n+1}$$

Similarly, $\mathrm{Lin}(f) = (\mathbb{Z}/d)^{\oplus(n+2)} \rtimes S_{n+2}$

$$|\mathrm{Lin}(f)| = (n+2)!\, d^{n+2}$$

General Idea of the proof

$X := \{f=0\} \subset \mathbb{P}^{n+1}$

$G := \mathrm{Lin}(f)$.

(1) Find a general upper bound for $|G|$

(2) $n \geq 25$: the Fermat hypersurf. has the largest automorphism gp of any sm. hypersurface X if $\dim(X)$ is large enough.

(3) $n < 25$

$\rightsquigarrow$ 80 exceptional cases where $|\mathrm{Aut}(X)|$ could theoretically exceed $|\mathrm{Aut}(X_F)|$

Show all but 8 of these 80 cases can be eliminated.

(1) Find a general upper bound for $|G| = |\mathrm{Lin}(f)|$

<u>Idea</u>: Decompose $\mathrm{Lin}(f)$ as a subnormal series and we bound each component individually.

$X \subseteq \mathbb{P}^{n+1}$ and $\mathbb{P}^{n+1} = \mathbb{P}(V)$ where $V \cong \mathbb{C}^N$ of dim $N = n+2$.

Rep. $G \hookrightarrow GL(V)$ of $G = \mathrm{Lin}(f)$ on V.

An irreducible representation of G is <u>primitive</u> if the underlying vector space V is not a direct sum of proper subspaces of V permuted under the action of G.

We call G <u>primitive</u> if $G \leq GL_N(\mathbb{C})$ and the corresp. rep. is primitive.

Lemma [Collins] For any subgp $G < GL(V)$, there is a direct sum decomposition

$$V = V_1 \oplus \cdots \oplus V_r$$

such that

(1) G permutes the subspaces $V_1, \ldots, V_r$, and

(2) $\forall\, i = 1, \ldots, r$, the group $H_i = \mathrm{Stab}_G(V_i)$ acts primitively on V_i.

Call $V_1 \oplus \cdots \oplus V_r$ a primitive decomposition of G.

Fix a prim. decomp. $V = V_1 \oplus \cdots \oplus V_r$ of G

$\updownarrow$

a dimension partition $\pi = (N^{\mu_N}, \ldots, 1^{\mu_1})$ of N, where

μ_k = # of times a subspace of dim. k appears in the decomp.,

$r = \mu_1 + \cdots + \mu_N$.

Define $H = \bigcap_{i=1}^{r} H_i$ $\qquad H_i := \mathrm{Stab}_G(V_i)$

= the kernel of the permutation action of G on V_i.

$\rightsquigarrow$ embedding of G/H in S_r and we have a subnormal series

for $G = \mathrm{Lin}(f)$:

$$1 \trianglelefteq Z(H) \trianglelefteq H \trianglelefteq G$$

<u>Ex</u> $X_F = \{f=0\}$, $f = x_0^d + x_1^d + \cdots + x_{n+1}^d = 0$

$V = \mathrm{span}\{e_0, e_1, \ldots, e_{n+1}\}$ and $V_i = \mathrm{span}\{e_i\}$, $i = 0, 1, \ldots, n+1$

$\Rightarrow V = V_0 \oplus V_1 \oplus \cdots \oplus V_{n+1}$ a prim. decomp.

$\Rightarrow \pi = (1^{n+2}) = (1^N)$

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<u>Idea</u> To find an upper bound for $|\mathrm{Lin}(f)|$ we bound

<u>Key tools</u>

G/H

$H/Z(H)$

a "replacement theorem" of Collins

$Z(H)$

(i) Show $Z(H)$ is a block scalar subgp of H;

(ii) Thm [Esser, L.] In our setup, the intersection of $\mathrm{Lin}(f)$ with the group of block scalar matrices has order at most d^n.

<u>Thm</u> [Collins]

$G \subset GL_N(\mathbb{C})$ · primitive subgp, $N \geq 2$

Then $[G:Z(G)]$ is bounded for fixed N. Moreover, if $[G:Z(G)]$ achieves this bound, then $G/Z(G) \cong S_{N+1}$ unless $\underbrace{N = 2,3,4,5,6,7,8,9 \text{ or } 12}$.

In these cases, the bounds are shown in table.
and if the bound is achieved

$G/Z(G) \cong H/Z(H)$

with

N	Largest $[G:Z(G)]$	H
2	60	$2.A_5$
3	360	$3.A_6$
4	25920	$\mathrm{Sp}_4(3)$
5	25920	$\mathrm{PSp}_4(3)$
6	6531840	$6_1.\mathrm{PSU}_4(3).2_2$
7	1451520	$\mathrm{Sp}_6(2)$
8	348364800	$2.O_8^+(2).2$
9	4199040	$3^{1+4}.\mathrm{Sp}_4(3)$
12	448345497600	$6.\mathrm{Suz}$

Let $\textcircled{H}(N)$ denote the function that returns the upper bound on $[G : Z(G)]$ over all primitive subgroups G of $GL_N(\mathbb{C})$. Then

$$\textcircled{H}(1) = 1 \quad \text{and} \quad \textcircled{H}(N) = (N+1)! \quad \text{except } N = 2-9, 12$$

<u>Thm</u> [Esser, L.] Using the quotient bounds,

$$|\mathrm{Lin}(f)| = [G:H][H:Z(H)]\,|Z(H)|$$

$$\leq \underbrace{(\mu_1!\,\mu_2!\cdots\mu_N!)\cdot\prod_{i=1}^{r}\textcircled{H}(\dim(V_i))\cdot d^r}_{B(\pi,d)}$$

<u>Ex</u> $X_F : \{f=0\}$

$$G = \mathrm{Lin}(f) = (\mathbb{Z}/d)^{\oplus(n+2)} \rtimes S_{n+2}$$

$$H = (\mathbb{Z}/d)^{\oplus(n+2)}$$

$$= Z(H)$$

Thm [Esser, L.] Fix $N \geq 27$ and degree $d \geq 3$. Then

$$B(\pi, d) < B((1^N), d)$$

for any partition π of N s.t. $\pi \neq (1^N)$.

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If $|\mathrm{Aut}(X)| > |\mathrm{Aut}(X_F)|$ of same dim. and deg., means $G = \mathrm{Lin}(F)$ has π with $B(\pi, d) \geq B((1^N), d)$.

$\Rightarrow N < 27 \quad (n < 25)$

$\rightsquigarrow$ 80 such cases "exceptional partitions" for degree d.

Number	N	Partition	Max d	B/B_F
No. 1	2	(2)	30	10.0
No. 2	3	(3)	7	6.66
No. 3	3	$(2,1)$	10	3.33
No. 4	4	(4)	10	40.0
No. 5	4	$(3,1)$	3	1.67
No. 6	4	(2^2)	17	33.3
No. 7	4	$(2,1^2)$	5	1.67
No. 8	5	(5)	3	2.67
No. 9	5	$(4,1)$	6	8.00
No. 10	5	$(3,2)$	5	6.67
No. 11	5	$(2^2,1)$	7	6.67
No. 12	5	$(2,1^3)$	3	1.00
No. 13	6	(6)	6	37.3
No. 14	6	$(4,2)$	6	26.7
No. 15	6	$(4,1^2)$	4	2.67
No. 16	6	(3^2)	4	4.45
No. 17	6	$(3,2,1)$	3	1.11
No. 18	6	(2^3)	12	66.7
No. 19	6	$(2^2,1^2)$	4	2.22
No. 20	7	$(6,1)$	4	5.33
No. 21	7	$(5,2)$	3	1.27
No. 22	7	$(4,3)$	4	7.62
No. 23	7	$(4,2,1)$	4	3.81
No. 24	7	$(4,1^3)$	3	1.14
No. 25	7	$(3,2^2)$	4	6.35
No. 26	7	$(2^3,1)$	6	9.53
No. 27	8	(8)	3	3.95
No. 28	8	$(6,2)$	4	13.3
No. 29	8	$(6,1^2)$	3	1.33
No. 30	8	(4^2)	5	45.7
No. 31	8	$(4,2^2)$	5	19.0
No. 32	8	$(3^2,2)$	3	1.59
No. 33	8	(2^4)	9	95.2
No. 34	8	$(2^3,1^2)$	4	2.38
No. 35	9	$(6,3)$	3	2.96
No. 36	9	$(6,2,1)$	3	1.48
No. 37	9	$(4^2,1)$	3	5.08
No. 38	9	$(4,3,2)$	3	2.12
No. 39	9	$(4,2^2,1)$	3	2.12
No. 40	9	(3^3)	3	1.06
No. 41	9	$(3,2^3)$	4	5.29
No. 42	9	$(2^4,1)$	5	10.6
No. 43	10	$(6,4)$	3	7.11
No. 44	10	$(6,2^2)$	3	5.93
No. 45	10	$(4^2,2)$	4	10.2
No. 46	10	$(4^2,1^2)$	3	1.02
No. 47	10	$(4,2^3)$	4	12.7
No. 48	10	(2^5)	7	106
No. 49	10	$(2^4,1^2)$	3	2.12
No. 50	11	$(4^2,3)$	3	1.85
No. 51	11	$(4,2^3,1)$	3	1.15
No. 52	11	$(3,2^4)$	3	3.85
No. 53	11	$(2^5,1)$	4	9.62
No. 54	12	(6^2)	3	3.02
No. 55	12	$(6,4,2)$	3	1.08
No. 56	12	$(6,2^3)$	3	2.70
No. 57	12	(4^3)	3	11.1
No. 58	12	$(4^2,2^2)$	3	3.08
No. 59	12	$(4,2^4)$	4	7.70
No. 60	12	(2^6)	6	96.2
No. 61	12	$(2^5,1^2)$	3	1.60
No. 62	13	$(3,2^5)$	3	2.47
No. 63	13	$(2^6,1)$	4	7.40
No. 64	14	$(6,2^4)$	3	1.18
No. 65	14	$(4^3,2)$	3	1.22
No. 66	14	$(4^2,2^3)$	3	1.01
No. 67	14	$(4,2^5)$	3	4.23
No. 68	14	(2^7)	5	74.0
No. 69	14	$(2^6,1^2)$	3	1.06
No. 70	15	$(3,2^6)$	3	1.41
No. 71	15	$(2^7,1)$	3	4.93
No. 72	16	$(4,2^6)$	3	2.11
No. 73	16	(2^8)	4	49.3
No. 74	17	$(2^8,1)$	3	2.90
No. 75	18	(2^9)	4	29.0
No. 76	19	$(2^9,1)$	3	1.53
No. 77	20	(2^{10})	3	15.3
No. 78	22	(2^{11})	3	7.27
No. 79	24	(2^{12})	3	3.16
No. 80	26	(2^{13})	3	1.27

Ex $(n,d) = (2,12)$

Let $X : \{f=0\} \subset \mathbb{P}^3$, $f = x_0 x_1 (x_0^{10} + 11 x_0^5 x_1^5 - x_1^{10}) + x_2 x_3 (x_2^{10} + 11 x_2^5 x_3^5 - x_3^{10}) = 0$.

$G = \mathrm{Lin}(f) = (12.A_5)^{\oplus 2} \rtimes S_2$ $\quad |G| = (12 \cdot 60)^2 \cdot 2!$

$H = (12.A_5)^{\oplus 2}$

$|H/Z(H)| = 60^2$

$Z(H) = (\mathbb{Z}/12)^{\oplus 2}$ $\quad |Z(H)| = 12^2$

Partition $\pi = (2^2) = (2^2, 1^0)$;

$\mu_2 = 2$ and $\mu_1 = 0$

$$B(\pi, d) = B((2^2), 12)$$
$$= 0!\, 2! \cdot \prod_{i=1}^{2} \Theta(\dim V_i) \cdot 12^2$$
$$= 2 \cdot 60^2 \cdot 12^2$$
$$= |G|$$

(2), (3), (4)

G/H

(1^N) $\quad H = Z(H)$